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A UNIFIED APPROACH TO DEVELOPING INTUITION IN MATHEMATICS

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Abstract

In this paper, we argue that creative insight in mathematics and the understanding of mathematics have in common a simple, highly integrated, creative mechanics involving both intellectual analysis and intuitive insight. Further, many of the problems currently facing mathematics education can be traced to the inability of educators to directly address the crucial step in this process, the spontaneous step of illumination, in which the mind integrates and expands new information and existing mental structures into a whole new mental structure and one directly grasps a solution or understands a proof. We analyze this mechanics in the light of the model of the mind described by Maharishi Mahesh Yogi's Vedic Science, and we suggest that the experience of transcending during the practice of the Transcendental Meditation technique trains the mind in this crucial step, thus completing the processes by which we learn and do mathematics and making the learning and doing of mathematics a joyful, fulfilling experience.

INTRODUCTION

It is well known that mathematics education is in trouble. At the school level, the Second Mathematics Assessment of the National Assessment of Educational Progress [**4**, p. 134] found that for the 9-year-olds mathematics was the best liked of 5 academic subjects, for the 13-year-olds it was the second best liked, and for the 17-year-olds it was the least liked subject. At university level many are unable to pass even first year calculus, and, of those who do, few would admit to enjoying the subject. Many reasons have been cited: lack of teachers interested in mathematics, lack of understanding of cognitive processes involved in learning and doing mathematics, teaching concepts before the student's cognitive development is ready for them, and so on. Even if one completes a degree in mathematics, one has, according to Littlewood [**28**], the "agony of research" to look forward to: a life spent mostly in frustration punctuated by rare inspirations. Contrast this scenario with the following experience.

In 1985 I had the good fortune to visit the Department of Mathematics at Maharishi International University, a nonsectarian private university in Fairfield, Iowa, with a strongly conservative curriculum, employing a system of education founded on the principles of Maharishi Mahesh Yogi's Vedic Science. I taught courses in calculus, linear algebra, analysis, and differential equations. It turned out to be the most rewarding and enjoyable teaching experience I have ever had. I found my students happy, bright, awake, interested in what I was saying, fearless, and enjoyed doing mathematics. Furthermore, they seemed to learn more easily, and to display a better intuitive grasp of new concepts and more creative approaches to problem solving than students I had taught elsewhere.

My experience at MIU changed my thinking dramatically about what it is possible for mathematics education to accomplish. In this talk I would like to share with you my views on the reasons for the enormous discrepancy between my experience at MIU and the general experience elsewhere, and to suggest that the methods used at MIU be adopted elsewhere.

My basic thesis is that mathematical knowledge progresses and is understood and learned through a common, simple, highly integrated creative process involving both intuitive insight and intellectual analysis. Most parts of this process have been successfully analyzed and taught, but a crucial step in the part of this process that pertains to gaining intuitive insight has so far evaded our efforts to teach it. This is the spontaneous step of illumination, in which the mind is able to cognize deep connections between its own structures and new information, putting them together into a new whole. At MIU, everyone practices Maharishi's Transcendental Meditation technique (TM), in which the active thinking mind spontaneously settles down to experience transcendental consciousness, a state of restful alertness in which awareness is only aware of itself. It is my view that regular experience of this process of transcending trains the mind in this crucial intuitive step, thus completing the basic creative process through which we learn and do mathematics. This step is also the step of the creative process that makes mathematics a joyful and fulfilling experience, which would explain my experience that students at MIU not only succeed at mathematics but also enjoy it.

In this paper, we will discuss the common features of the creative process and the process of understanding, bringing out the important role of intuition and of schemas (or intuitive biases) and pointing out the weak link. We will explain how experience of transcendental consciousness strengthens not only this crucial link, but also other aspects of the creative process that often go wrong, thus completing the creative process and making it always accessible to students and mathematicians.

In addition to the practice of the TM technique, a second aspect of the approach at MIU is the drawing of parallels between principles of mathematics presented in class and more universal principles common to all disciplines and to the growth of human consciousness. These principles, which are drawn from the body of knowledge known as Maharishi's Vedic Science, appear to provide a valuable source of insight into mathematical processes, functioning as a body of established schemas, or intuitions.

The paper is presented in the following sections:

1. The role of intuition in modern mathematics.
2. Solving current problems in mathematics education.
3. Research into the effect of Maharishi's TM and TM-Sidhi program on factors affecting success in mathematics.

1. THE ROLE OF INTUITION IN MODERN MATHEMATICS

The importance of intuition, both in research and in education, is probably appreciated more in mathematics than in any other field. In the preface to *Geometry and the Imagination*, Hilbert and Cohn-Vossen describe the two complementary tendencies in mathematics:

> In mathematics, as in any scientific research, we find two tendencies present. On the one hand, the tendency toward abstraction seeks to crystallize the logical relations inherent in the maze of material that is being studied, and to correlate the material in a systematic and orderly manner. On the other hand, the tendency toward intuitive understanding fosters a more immediate grasp of the objects one studies, a live rapport with them, so to speak, which stresses the concrete meaning of their relations [**23**, p. iii].

Poincaré refers to the role of intuition in integrating parts into a whole:

> In the edifices built up by our masters of what use is it to admire the work of the mason if we cannot comprehend the plan of the architect? Now pure logic cannot give us the appreciation of the total effect; this we must ask of the intuition [**36**, p. 217].

The Two-Fold Process of Acquiring Mathematical Knowledge

All levels of mathematical knowledge, both informal and formal, both concrete and abstract, progress by means of a deeply integrated process of reflective, analytic thinking and intuitive thinking. On the one hand, analytic thinking emphasizes precision, objectivity, logic, symbols, operations, rigor, details. It aims to arrange accumulated knowledge in a rigorous, logically sound, deductive sequence of steps to arrive at safe conclusions. It tends to be highly reliable. One thinks one has some idea how to teach it. It seems to fit into current models of human cognitive development, particularly Piaget's endstage of adolescent development, formal operations. On the other hand, intuitive thinking is subjective, immediate, global, nonverbal, synthetic. "Such a cognition is felt by the subject to be self-evident, self-consistent, and hardly questionable" (Fischbein & Gazit [**17**, p. 2]). Compared to analytic knowledge, intuitive knowledge is generally held to be unreliable. Skill in it can only be sought, not taught. And modern psychology offers no single comprehensive model of cognitive development that encompasses it.

Let's examine some of the ways in which these two processes, subjective and objective, are interwoven. On the one hand, after an initial stage of exploration involving both intuitive and analytic steps, the solution to a mathematical problem usually occurs to one as a sudden intuitive grasp. Because of its global subjective nature, an intuitively grasped cognition then needs to be crystallized and brought out by means of reflective thinking into the realm of mathematical discourse. Usually one does this by placing it in the context of first informal, then formal theory (see, e.g., Wittman [**49**] for a useful discussion of the role of informal mathematics). This procedure also serves to verify the correctness of intuitions by testing them against the rules of deductive logic, our main criterion for the objective correctness of mathematical knowledge.

On the other hand, intuition is used even in this process of bringing intuitive knowledge into the formal structure of mathematics. For example, when constructing a rigorous proof, we use intuitive interpretations, geometric insight, diagrams, analogies, particular concrete examples from our experience. Afterwards, intuition may serve as a check on the correctness of the formal proof that we have constructed. For example, Wittman points out that it is intuition that makes us suspect there is an error in the "proof" of the assertion that all triangles are isosceles. On the basis of this intuition, one then discovers that the diagram used is faulty. A recent conference of the Humanistic Mathematics Network [**48**] stressed the role of intuition, not only in understanding, but in creating concepts that appear in their finished versions to be "merely technical." In creating new mathematics, the investigator "as in every other science, does not work in this rigorous deductive fashion. On the contrary, he makes essential use of his imagination and proceeds inductively aided by heuristic expedients" (Felix Klein, quoted in [**27**, p. 274]).

As mathematics educators, most of us appreciate the need to use an integrated approach in the classroom. Petitto [**34**] studied strategies used by 9th grade students to solve algebraic equations. She found two groups: one that leaned towards an intuitive global approach, trying to capture numerical relationships among numbers in an equation without transforming the equation itself, and one that relied on memorized or routine step-by-step procedures for transforming the equation and producing the answer. Those that moved easily between the two proved most successful. Wittman [**49**] proposes that the main task of mathematics teaching at the school level is the development of informal mathematics. He advocates practice in intuitive, operative studies of a rich variety of examples and models, in discovering patterns and constructions through one's own activities, and then reflecting on these intuitive activities by finding general formulations, finding proof ideas for the patterns and constructions, and so on. This he regards as prerequisite to axiomatics, to formal mathematics. Fischbein's comprehensive studies of the role of intuition in learning (below) also bring out the crucial integration of intellectual and intuitive understanding.

In the field of science in general, Hanson [**21**] has argued that all knowledge, whether gained by subjective or objective means, is based in intuition.

Let us now turn to the concept of intuition itself.

The Nature of Intuition

Intuition appears in many different guises. In mathematics it seems to function in two main processes. The first commonly comes up in problem solving or mathematical research. It is the classic intuitive leap or sudden insight described by Poincaré [**37**], Littlewood [**28**], and others. The process is usually held to involve four steps. Following Littlewood's description, the first is a period of preparation in which one strips the problem of accidentals and brings it clearly into view, surveys all the relevant knowledge, ponders possible analogues. Newton, he says, suggested keeping it constantly before the mind during intervals of other work. The second is a period of incubation, which could last a moment or several years, and during which one may not be consciously thinking about the problem at all. The third, called illumination, often happens in a fraction of a second, when the creative idea is suddenly grasped in its wholeness, before any explicit details and justification are given. The fourth then involves working the idea out, the reflective activity of verifying its truth and integrating it into formal mathematics. This experience of creative insight may take place briefly when solving a problem in class, or may take place over a period of years when a practicing mathematician discovers the solution to a research problem.

Secondly, we apply the term intuitive to an explanation or interpretation directly accepted as natural, self-evident, intrinsically meaningful. For example, Fischbein [**14**] points out that most people find the equality of opposite angles formed by two intersecting lines highly intuitively obvious, but find the fact that the sum of the angles of a triangle is equal to two right angles not intuitively obvious. Fischbein [**14**], [**16**], [**17**] has written extensively on the need to provide an intuitively acceptable explanation of a proof or to provide a "basis for belief" based on active personal involvement along with or prior to a formal step-by-step logical argument. Without this more holistic direct feeling for the correctness of a proof, he finds, students tend not to be able to remember the proof nor to effectively use it later. A related idea is Skemp's "learning with understanding." In mathematics, Skemp says, agreement depends on pure reason, not on authority. Therefore, one must learn mathematics with understanding, not by rote. One must establish mental patterns, or schemas, rather than memorize facts. Only then will the knowledge be useful later on.

The concept of schemas is fundamental to Piaget's model of cognitive development [**35**]. One of the most powerful tools of the human mind is that of conceptualization, the ability to abstract a concept from a list of examples displaying that concept, to treat that concept as an entity in itself, perform operations on it, compare it with other concepts, and so on. This ability is fundamental to mathematics. A conceptual mental structure, a body of concepts interrelated in some way, is often called a schema. For example, a student builds up a collection of concepts involving numbers and some relationships between them. His conception of how they all fit together and are used is a familiar mental schema approximating what we call arithmetic. Later on when some problem requires addition, he spontaneously evokes this schema and is able to find the sum. Later in his mathematical career he may see this same problem in the context of his understanding of group theory, another schema. Schemas may be very local or very comprehensive, very simple or very complex. They may involve geometry or algebra or everyday life. A single example may invoke many different schemas, from which one must choose, either automatically or consciously. Once in place in the mind they seem to be invoked automatically when the need arises. (See Skemp's book [**43**] for a clear introduction to mathematical schemas.)

According to this model, what makes a mathematical explanation feel natural and intuitively acceptable, or a problem easy to solve, is that it fits into, easily extends, or easily integrates our previously established mental schemas. These mental schemas may already be in place, coming from one's stage of cognitive development or from previous experience, mathematical or otherwise, or they may be put in place by a teacher who gives an intuitive justification of a proof before giving the formal proof, or who provides examples and informal experience before introducing an abstract concept or definition. The more schemas one has available, the better the chance of coping with new knowledge.

The notion of having schemas in place is also useful in understanding the nature of mathematical research. This concept of schematic structures appears to correspond to Dieudonné's conception of intuitions [**8**]. He held that intuition regarding mathematical objects is acquired by growing experience and that there are different intuitions in different fields, geometric intuition, combinatorial intuition, and so on. Progress in modern mathematics was achieved essentially by transfer of intuition from one mathematical field to another, for example, linear algebra came about as a transfer of geometric intuition to algebra. According to Dieudonne, the more schematically and abstractly mathematical objects are comprehended, the more deeply relationships can be understood and transferred intuitively. Hilbert was a living example of this principle. He mastered one field of mathematics after the other, leaving one field for the next just as he had reached his peak of success. In this way he was able to make profound use of transfer of intuition from one field to another.

At first glance these two processes, creative insight and the combining of intuitive acceptance with formal mathematics to produce integrated understanding, may appear unrelated. But they have key aspects in common. Both involve a preparatory stage that serves to enliven the mathematical content and the intention to solve the problem or understand the explanation. Both evoke established schemas, or mental structures. Both processes involve a crucial step of integration and expansion of knowledge and schemas into a new more comprehensive whole. This move from parts to whole involves letting the attention disengage itself from its identification with the content and activity of the preparatory stage, thus leaving the mind lively in the problem but free to settle down. In this lively, free state the mind, if it is going to, will spontaneously settle down and directly grasp the whole. This whole may be experienced as the solution to a problem, or as a sharp concept that was being sought, or as an understanding of a proof or explanation. As a result of this step, the new integrated whole becomes available to intellectual analysis and verification, and ultimately may become a schema for future use in further cycles of the creative process and understanding.

The step in which the mind settles down and an integrated whole dawns in the awareness is invariably a blissful experience, often called the "aha" experience. Because of its direct, spontaneous nature it is called intuition . In fact, because of its immediacy, Gödel [**46**, pp. 84–86], Littlewood [**28**, p. 113], Polanyi [**38**, p. 118], Fischbein [**14**], and others have related this sudden direct grasp to sense perception. It is the union of the intellectual understanding of a mathematical truth (for example, a proof or concept) with a more immediate, global feeling for that truth that produces the joy of mathematics and makes doing mathematics fulfilling.

It is this union that Fischbein says is so necessary for real understanding. In his paper on intuition and proof, Fischbein explains, "a mathematical truth can become really effective for productive mathematical activity if, together with a formal understanding of the respective truth, we can produce that kind of synthetic, sympathetic, direct acceptability of its validity. The same holds for a concept, a statement, a proof, and for the basic principles of generalizing, deducing, and proving in mathematics" [**14**, p. 18]. Sometimes this is easy. For example, the equality of opposite angles where two straight lines intersect. Sometimes there is no simple way, but Fischbein holds that such a fusion makes the knowledge satisfying, acceptable, remembered. One needs a *feeling* that the proof is right, not just a memory of the steps of the proof.

Poincaré [**37**] connected the creative process with the faculty of feeling when he said that mathematical creation requires strength of memory (but not prodigious), attention, and a delicate *feeling*, an intuition of mathematical order, that makes us divine hidden harmonies and relations.

In the light of these deep parallels between the creative process and the process of integrated understanding, it is my view that these two apparently unrelated processes are really expressions of a common underlying creative mechanics. Its key is the ability to allow the mind to leave the superficial active level of thinking about the problem or explanation and spontaneously settle to a level where intellect and feeling are united, where new information and established schemas are integrated and expanded, where intellectual knowledge becomes immediate.

It is my view that this type of process permeates every level of doing and learning mathematics. The process, however, has eluded our efforts to teach it. We have addressed much effort to the preparatory stage, teaching heuristics, trying to set up appropriate schemas in the education system, trying to relate new knowledge to old as we teach, and so on. Many of us have emphasized informal mathematics, in which we structure into our classes opportunities for the creative, integrative process to take place. And all of this has helped to some degree, but we have been unable to directly address the integrative step itself, in which the mind spontaneously settles down and puts everything together, uniting intellect and feeling to create true understanding. It is this crucial step that regular practice of the Transcendental Meditation technique addresses directly. The TM technique provides the basis for completing the creative integrative process required for mathematics to progress. Once this process can be experienced in its completeness, once the spontaneous integration of intellect and feeling is always available to solving and understanding, new schemas, or mental structures, are established with least effort, deeply, and are available for the next step of mathematical education or research. The knowledge is part of the student or mathematician.

Let us now look at the way in which the TM technique fills this gap and helps to solve the problems of mathematics education.

2. SOLVING CURRENT PROBLEMS IN MATHEMATICS EDUCATION

The experience of mathematicians and mathematics educators, that formal intellectual understanding and intuitive grasp must be united in order to produce the creative process and real understanding, can be viewed in the light of the model of the mind presented by Maharishi's Vedic Science. According to Maharishi [**31**], the mind is structured hierarchically in increasingly abstract, powerful, and comprehensive levels: senses; active thinking mind (the associative faculty, including apprehending and comparing); intellect (the abstract discriminative faculty including analysis and synthesis); subtle feeling (responsible for intuition), which is usually said to be the most refined and subtle level of intellect; ego (the most subtle integrative function of individuality); and Self (a completely abstract level also called pure awareness or pure consciousness). Note that subtle feeling, or intuition, is considered in this model to be the most refined and powerful level of intellect, and that when the mind functions from this level, intellect and feeling are united as an integrated means of gaining knowledge.

Maharishi further describes the TM technique as a process whereby the attention is spontaneously and naturally freed from mental and physical objects of perception and allowed to settle to more and more subtle and comprehensive levels until it transcends thought altogether and one arrives at one's own simplest, most integrated and comprehensive state of awareness, a state of restful alertness in which awareness is aware only of itelf. This is the level of mind called Self or pure consciousness, and the experience is called transcendental consciousness. This experience is said to enliven the more subtle and powerful levels of the mind, thus enhancing creativity and allowing one to use one's full mental potential in all areas of life. (See Section 3 for research supporting this.)

Maharishi speaks of the role of education in developing in students the ability to spontaneously think deeply on quieter levels of the mind, where knowledge is stored (Lecture, May 5, 1971). He explains that as one's attention goes to the quieter levels of the mind, the quiet, tender, comprehensive level of the mind is enlivened, where deep connections are made. The purpose of education, he says, is to enliven those quiet levels.

Further, Maharishi explains that regular experience of transcending during the TM technique results in growth to higher states of consciousness, each of which presents even more powerful direct means of gaining knowledge. These higher states would be recognized from a Piagetian viewpoint as truly post-representational stages of cognitive development, beyond formal operations. Evidence that practice of the TM technique is helpful in stabilizing Piagetian stages of cognitive development up to and including formal operations is cited in Section 3. See also [**2**] for a survey of research suggesting that it promotes development beyond formal operations. A discussion of the full range of human development to higher states of consciousness is beyond the scope of this paper. See [**1**] and [**2**] for a discussion of the way in which the model of cognitive development presented by Maharishi's Vedic Science provides a single comprehensive model of cognitive development and a comparison of this model to current psychological theories of cognitive development up to and beyond formal operations.

By promoting growth of successively more comprehensive and powerful stages of cognitive development, regular experience of transcendental consciousness is clearly a most valuable asset in the study and doing of mathematics. We will not extend this paper into this vast area, however. Let us look instead at how, by itself, experience of transcendental consciousness is the basis for solving current problems in mathematics education.

Completing the creative process

Experiencing transcendental consciousness regularly enables the key aspect in creative insight and in mathematical understanding to happen: the crucial step of deep integration and expansion of knowledge and schemas into a new more comprehensive whole, described earlier.

It does this by promoting the habit and ability of letting the attention settle to deeper, more comprehensive levels of the mind. The spontaneous, natural process of allowing the attention to free itself from mental or physical objects of perception and allowing that mental activity to naturally settle down, hence naturally settle to the most comprehensive and integrated level of all, transcendental consciousness, becomes a familiar process. Regular practice twice daily of Transcendental Meditation allows this to happen naturally and regularly. As this process becomes more familiar, it becomes readily available during daily activity. Thus if we are working on a problem in class, or working on a research problem, or just listening to a proof of a theorem, the mind will take the opportunity to settle down and experience deeper levels, so that the step of illumination, when we grasp the whole of whatever it was we were thinking about, will take place. Without this habit and without a state of relaxed alertness, when trying to solve a problem or understand a proof, we tend to try to keep thinking about it, which prevents the mind from settling to deeper levels. Even trying to think about the possible wholeness does not help, because that is still holding the mind on the active level of involvement with thoughts. The experience of wholeness occurs spontaneously when the attention is freed from thoughts and is able to go within to a deeper level of the mind. This is what regular experience of transcending promotes.

As research mathematicians, Littlewood and Poincaré knew the value of letting a problem go. They both structured into their lives time for allowing the attention to disengage from the level of actively thinking about a problem. Littlewood went for walks in the country. Poincaré went on a geology excursion. And occasionally, with good luck, the sudden flash of deep inspiration occurred.

In my opinion we can now take the element of luck out of the process. We can directly train students or ourselves in the ability to take that intuitive step of illumination or understanding, simply by teaching students the practice of the TM technique. This would make our efforts to teach heuristics and to structure schemas at the right time effective and fruitful. With the completion of the creative process, students should be able to solve problems significantly better than before and be able to learn, remember, and use new knowledge. Quite an extensive body of research suggests that this should be the case (see Section 3). Further, research appears to support the view that the TM technique provides this ability to use the quieter, more comprehensive levels of the mind spontaneously at any time. In tests of field independence, students have been found to grow in the ability to focus sharply and maintain broad awareness (see, e.g., [**33**], [**9**]) possibly indicating growth to the stage of development where the mind permanently functions from its deepest, most comprehensive level.

Alleviation of math anxiety and other affective factors

Other factors affecting the creative process are also affected by experience of transcendental consciousness. Both the ability to focus and the ability to settle to a deeper, more comprehensive level of the mind require a state of relaxed alertness. Stress and anxiety interfere with this. It is well known that higher mental activities are the first to be affected by stress and anxiety (see, e.g., Skemp, [**43**, p. 118]). Negative views about one's ability and about the difficulty of doing mathematics also affect the creative process. They create anxiety and also may prevent one from even beginning the preparatory stage. Practice of the TM technique has been found to provide deep rest to the nervous system and to relieve anxiety and stress. (For a comprehensive survey of physiological research on the TM technique, see [**11**].) It also increases self-confidence and self-actualization (see Section 3). This in itself should decrease math anxiety and improve confidence in doing mathematics. With growing relief from anxiety and stress, the state of relaxed alertness experienced during TM becomes more and more a part of everyday life, always there even in dynamic activity, hence always available to the creative process. With the complete creative process available, a student will find himself actually able to do mathematics. Success is also good for overcoming math anxiety.

Overcoming the feeling that mathematics is remote

It is widely held that one reason potentially good students do not pursue mathematics as a career is that they find it remote from personal experience. Intuition, as we have pointed out, is a crucial step in creating and understanding mathematics. In order to bring intuitive knowledge into the mainstream of mathematical discourse, however, we embed it in the framework of formal theory, using deductive logic as our criterion of right knowledge. This approach has made mathematics a body of reliable, irrefutable, useful, and elegant knowledge. But at the same time its very success depends on removing any element of ourselves from the knowledge. Further, from the viewpoint of Piagetian cognitive development, a strict diet of logical formal theory in the classroom may lock one into the stage of formal operations, which Cook-Greuter [**6**] has called a stage of maximum distance between thinker and object of thought. Formal operations is the stage of the reflective intellect. It allows the thinker to deal only indirectly with objects of thought, through mental representations of them. Even the self becomes one more object to reflect on indirectly through representation. And as Maharishi has said: "When the knower does not know himself, then all knowledge is baseless. And baseless knowledge can only be nonfulfilling" (Lecture, August 1, 1986). Since formal operations begins to stabilize around age 12, perhaps there is some relationship between this side effect and the growing distaste for mathematics that students exhibit from ages 13 to 17, which we referred to in our introduction.

One way we as teachers try to overcome the feeling of remoteness in our students is by relating the mathematical theory to applications, which, it is hoped, mean something to the student. This is useful, but does not address the above much deeper problem. What is required is to overcome the intrinsic separation of student from mathematics imposed by representation itself. This is what is achieved in the step of illumination in the creative process and in the process of understanding, in which the whole is grasped directly before any representation or intellectual analysis can be made.

Ensuring correct schemas are in place beforehand helps facilitate this integrative step. Also, teaching informal mathematics sets up many opportunities for these processes to happen. But in both cases these processes should be far more likely to happen, and deeper levels of integration should occur, when the ability to transcend is also available.

Students who have the direct experience of this profound process of integration of intellect and feeling regarding a mathematical truth several times in each class are not likely to feel mathematics is remote from them. The processes take place within them, and the mathematical knowledge produced by them is part of them, a mental structure they have uncovered within themselves. It is very personal knowledge.

At MIU students find mathematics personally relevant also because it is pointed out to them that the principles by which their own consciousness unfolds are the same as the principles by which mathematics unfolds. The body of knowledge concerned with the unfolding of consciousness is called Maharishi's Vedic Science. This aspect of MIU is explained in the next subsection. The point here is that, as Maharishi remarks, when one knows that mathematics is really nothing but the study of one's own intelligence, which has its source in one's own simplest state of awareness, transcendental consciousness, which one experiences every day, then the study of mathematics becomes completely intimate. Both transcending and Maharishi's Vedic Science make studying mathematics intimate and hence deeply fulfilling.

Further means by which the student is connected to mathematics at MIU are classroom charts that place the topics covered in the lesson in the context of the whole lesson, place the lesson in the context of mathematics as a whole, place mathematics in the context of the wholeness of all knowledge, and connect the wholeness of all knowledge to its basis in the student's own consciousness. Although students focus sharply on the mathematical content, they never lose sight of the most global perspective on the topic being taught, and never lose sight of themselves, as the source of the knowledge.

Making the most of schemas

New information or problems that do not fit easily into established schemas, or that do not easily extend existing schemas often cause conflicts that students are unable to resolve satisfactorily. Either students give up trying to understand mathematics, or they blindly apply wrong schemas to the new information. Both avenues lead to frustration.

Fischbein seeks the source of difficulties students encounter in the basic intuitive patterns they use when solving mathematical questions, that is, in their established schemas. His extensive and illuminating research into intuitive biases of primary and secondary students led him to the conclusion that "the problem of identifying the natural intuitive biases of the learner is important because they affect—sometimes in a very strong and stable manner—his concepts, his interpretations, his capacity to understand, to solve and to memorize in a certain area. We are naturally inclined to retain interpretations which suit these natural, intuitive biases, and to forget or to distort those which do not fit them" [**19**, p. 491]. Thus intuition, which gives us the creative process itself, may degenerate into blind rigidity when ones schemas are not correct.

Feller [**13**] and, more comprehensively, Fischbein [**16**] found children's natural intuitions about probability usually contradict the way probability actually works. Also, Fischbein and others [**18**] found in studies of children's intuition of infinity that conservation mechanisms from lower stages of cognitive development may be wrongly applied to problems concerning infinity.

Students also have inadequate schemas regarding formal mathematics itself. Fischbein [**14**] found intuitions regarding the nature of formal mathematics are often absent. High school students were not necessarily convinced of the truth of a statement simply by understanding a formal logical proof. Also to this point, Schoenfeld [**41**] gave students a geometry problem and found that most used pictures to try and solve it. None of them used theorems from Euclidean geometry. They did not realize that theorems and constructions of formal geometry could be used in solving a problem.

Another reason schemas may be absent at the right time in the course of education is that a student may not have reached the level of cognitive development in which such schemas are established. For example, Driscoll [**12**, pp. 19–24] claims many topics taught in school disadvantage students who have not yet reached the Piagetian stage of formal operations: formal proof, the concept that an equation remains the same when the variable is changed, fractions, conservation of length, volume, area, and word problems. He felt that some, but not all, of this could be overcome by appropriate teaching methods. In this regard Carpenter [**4**, p. 144] points out that mathematics instruction has been found to aid development and stabilization of new cognitive stages if it is done carefully, but it is rarely done carefully.

Skemp [**43**] advocates structuring the mathematics curriculum in such a way that when a topic is taught, the mental schemas that are needed for its understanding are already in place. He also points out that in the course of simplifying material so that it can be understood by elementary students, teachers must be very careful to give students schemas that are easily extendable at a later stage rather than schemas that have to be unlearnt. For example, children develop strong intuitions that addition and multiplication make something bigger and that division and subtraction make something smaller. This leads to trouble in first meeting negative numbers and fractions less than 1.

We can see that the problem of inadequate or wrong schemas is an important one. Identifying wrong schemas or intuitive biases and changing them is necessary. This is not always easy. Fischbein points out that to change a wrong schema, one has to entirely re-elaborate the process. Further verbal instruction does not help. It is my view that the TM technique should help here as well. As we suggested earlier, by culturing the ability to allow the mind to settle spontaneously to deeper, more integrating levels, practice of the TM technique promotes integrated intellectual and intuitive understanding of mathematical arguments. Thus understanding a re-elaboration at a deep level should be smooth and easy. Further, according to the model of cognitive development presented by Maharishi's Vedic Science, more comprehensive perspective develops as one is able to function from deeper, more comprehensive levels of the mind. This makes the mind more flexible, less rigid. Broader perspective should help students to let wrong intuitions go when they become aware of them, and even help them become aware of contradictions in their thinking themselves.

Let us now turn to establishing schemas.

Schemas (bases of belief, intuitive patterns, established mental structures) are an important ingredient in the creative process and the process of understanding, as we have brought out. Bad ones interfere with the process, good ones are important for success. Where do they come from?

They seem to be the result of past occurrences of the same creative integrative process we have been discussing. Indeed, if this process is our basic mechanism of assimilating knowledge and developing cognitively, one imagines a long sequence of recurrences of the process throughout life going back to assimilation of sensory-motor experience in childhood. At each step one directly experiences the new information integrating with the old mental structures to produce new structures.

Establishing a basis for belief, according to Fischbein [**14**], requires that one "live the process" (p. 14). One needs to be involved directly, personally, behaviorally, either mentally or physically. For example, Fischbein felt the reason students were not convinced of the truth of a theorem after accepting a formal proof of it was that "the concept of formal proof is completely outside the mainstream of behavior" (p. 17). In this scientific age, conviction that a conclusion is true is instead "derived from a multitude of practical findings that support the respective conclusion" (p. 11).

A most interesting example is the following.

One of Fischbein's fascinating projects has been the study of the intuitive patterns of primary and secondary students regarding mathematical concepts of infinity [**14**], [**15**], [**18**], [**19**]. Aristotle considered infinity only as a potentiality. For example, a line can be extended indefinitely or a segment can be divided indefinitely. Not until Cantor introduced infinite cardinal numbers did actual infinity become part of mathematics. In the studies cited above, students in grades 8 and 9 appeared to be comfortable with the notion of an indefinitely repeated process, potential infinity, but highly uncomfortable with actual infinity. Fischbein's explanation of this was that "actual infinity has no behavioral meaning and therefore is not congruent with an intuitive, insightful interpretation." That is, it is beyond any direct experience we could have, either mentally or physically. Further, studies of students in grades 5 to 9 suggested that development of concepts of infinity were tied to cognitive development. Up to age 12, roughly the beginning of the formal operational stage, intuitive interpretations of potential infinity were still in the process of formation. After age 12 intuition of infinity appeared to be stable, but was still based on finitist schemas. It appears that actual infinity is not available as a schema within the formal operational framework.

This raises some interesting questions. Is such a schema available to a person who has attained post-formal operational stages of development, for example, to Cantor? Are there mathematical truths for which one could never have a schema, that is, for which one could never have a direct intuitive feel?

We suggest that experience of transcending and development of higher states of consciousness vastly extends the range of direct experience from which we can draw, and that schemas based on this experience are fundamentally related to mathematics. Maharishi's Vedic Science is the science of consciousness itself. It studies principles governing all orderly change and growth. These principles are directly experienced in transcending during practice of the TM and the TM-Sidhi program or in our everyday life. According to Maharishi, the principles described by Vedic Science, governing the way consciousness unfolds, are the deep principles governing any discipline, including mathematics. Thus study of Maharishi's Vedic Science, together with direct experience of transcending, should establish in one a body of very deep schemas upon which intuitions of a mathematical nature can be based.

A comprehensive discussion of the principles of Maharishi's Vedic Science and how they relate to mathematics is beyond the scope of this paper. For an introduction to Maharishi's Vedic Science, see Maharishi Mahesh Yogi [**30**] and Chandler [**5**], and for a comprehensive discussion of deep relationships between Vedic Science and foundational areas of mathematics, see Weinless [**47**]. Briefly, here is an application to the intuition of actual infinity from Weinless's paper.

The experience of transcendental consciousness itself provides a behavioral basis for actual infinity. A completely non-representational state, awareness, by virtue of being aware, is aware of itself. This is not awareness thinking about a representative of itself because the active thinking mind has settled down to its ground state. It is direct experience of wakefulness, with no boundaries, unbounded awareness. Maharishi describes this state as the absolute. This corresponds to Cantor's description of the absolute infinite, which includes and surpasses all the different levels of the actual infinite in set theory.

According to Weinless, Maharishi's Vedic Science provides deep principles of the structure and functioning of intelligence that can unify our understanding of mathematics, and which, together with the experience of transcendental consciousness, can provide an intuitive basis for the deepest principles of mathematics.

3. RESEARCH INTO THE EFFECTS OF MAHARISHI'S TM AND TM-SIDHI PROGRAM ON FACTORS AFFECTING SUCCESS IN MATHEMATICS

There is an extensive literature on research into the effects of Maharishi's Transcendental Meditation and TM-Sidhi program on factors affecting education. For the sake of brevity, we shall summarize some of these, and refer the interested reader to the papers [**11**], [**2**], [**32**], [**29**], which provide in-depth surveys of the research behind these statements and many other studies. Although these factors affect education in general, many of them have been shown to affect mathematics education specifically. For example, field independence [**25**], [**40**], [**45**], intelligence [**24**], and self-concept [**39**] have been found to correlate positively with mathematics achievement, whereas field dependence [**22**] correlates with math anxiety, and building a positive and realistic self-concept can prevent math anxiety [**7**].

Cognitive, affective, and physiological characteristics that contribute to effective learning have been found to improve in students practicing the TM technique. Improvements have been found in alertness, memory, fluid intelligence [**3**], [**9**], [**42**], field independence [**9**], creativity [**42**], [**44**], reasoning ability [**10**], and academic achievement. There have also been many studies indicating improved self-concept, increased self-actualization, reduced depression, neuroticism, and anxiety, reduced aggression and increased tolerance, greater emotional stability, and greater physiological resistance to stress.

Implementing programs in TM and Maharishi's Vedic Science in existing classes has been successful as well. Findings are consistent with those above. For example, Shecter [**42**] found that when compared to non-meditating control groups, students in classes practicing the TM technique showed increases over a 14-week period in fluid intelligence, creativity, energy level, innovation, self-esteem and tolerance, and decreases in anxiety and conformity. This program has been introduced in all levels of education in over 20 countries, including Australia, Brazil, Denmark, Dominican Republic, Great Britain, The Netherlands, India, Kenya, Korea, Norway, Philippines, Puerto Rico, Taiwan, Thailand, and the US. Studies in India and in England, for example, found improved reading comprehension, memory and concentration, and improved grade-point average.

A number of studies have monitored the educational experience at MIU and at its affiliate, Maharishi School of the Age of Enlightenment (MSAE) (kindergarten through grade 12). Cross-sectional studies indicate that MIU students score higher than controls and norms on scales of self-actualization. Longitudinal studies of MIU undergraduates over four years found increases in fluid intelligence, in contrast to no change in normative trends [**3**], [**9**], increased field independence [**9**], and increased social maturity and psychological health as indicated by personality tests [**3**].

Longitudinal studies over shorter periods indicate that students at MIU who learn the more advanced TM-Sidhi program, compared with matched MIU students who practice only the TM program, show significantly increased abstract learning ability (concept learning), increased flexibility of the central nervous system (faster recovery of the paired Hoffman reflex), and increased orderliness of brain functioning (as indicated by EEG coherence in frontal brain areas)(see, e.g., [**10**]). These cognitive and neurophysiological developments occur together as an integrated whole [**10**] and higher levels of these traits predict higher academic performance.

It appears that their educational system enhances cognitive development (and hence mathematical ability) in MSAE students. In two different studies, children at MSAE performed better than non-meditating children elsewhere on standard Piagetian tasks in conservation. Secondary students at MSAE performed better on a test of creativity than control subjects taken at random from a normative data bank of the test. Length of time practicing the technique was significantly correlated with level of performance. Other studies found greater field independence on cognitive-perceptual tasks in meditating children ages 7–11 than matched controls. (See [**11**] for references.]

Despite the liberal admissions policy, academic achievement has been extremely high at MSAE. The majority of classes score in the top five percent nationally on the Iowa Test of Basic Skills (administered to grades K–8) and the Iowa Tests of Educational Development (administered to grades 9–12) and many classes score in the top one percent. Individual students show an increase in percentile level on the tests of language, reading, and mathematics over the course of one school year. Teachers and visitors to the school remark on the children's greater awareness, greater ability to focus, increased creativity, broader perspective, flexibility, and enjoyment of their studies, all of which are needed to succeed at mathematics.

CONCLUSION

Concerning the NCTM's current "decade of problem solving", Galovich said, "We seem to face the unhappy situation of wanting to do something to help our students become better problem solvers, taking what seem to be the obvious steps to bring about this goal, and discovering that our efforts are only marginally successful [**20**, p. 68]." The same is true of understanding mathematics, and doing mathematics generally. Bruce Vogeli of Columbia University Teachers College was quoted in Time magazine recently as calling innumeracy the "major untouched educational issue of the decade" [**26**, p. 66]. Why is the problem untouched in spite of all our efforts to improve mathematics teaching?

In this paper we suggest that the key to improving the doing and learning of mathematics is the addition of the regular experience of transcending, that is, of Maharishi's Transcendental Meditation program, to the present curriculum. We argue that this experience directly structures in students the ability to take the one step in the creative process and the process of assimilating new knowledge that none of our other approaches has been able to address directly: the intuitive step, whereby new information and existing mental structures are spontaneously synthesized and extended into whole new mental structures, the step of illumination. It is this step that enables the student to grow, to expand his mathematical awareness into a greater whole, and to experience the direct relationship between himself and mathematics. It is this step that makes mathematics a joy.

Transcending is also the experiential basis of fundamental principles of the unfoldment of consciousness embodied in Maharishi's Vedic Science. If we introduce lessons on Maharishi's Vedic Science along with practice of the Transcendental Meditation technique, then we are also able to address the problem of providing deep schemas for mathematical concepts, such as completed infinity, for which students, at least those who have not progressed beyond the cognitive stage of formal operations, have no experiential basis. This body of deep unifying principles governing the mechanics of consciousness may also provide a profound intuitive basis for research areas of mathematics.

It is a popular belief that there can be no one solution to all the seemingly diverse problems of mathematical education. Yet the concept of one crucial idea that makes a whole body of mathematics work—for example, the limit process in analysis—is the stuff mathematics is made of. Also central to mathematics is the idea that a single weak link in a proof can invalidate the entire proof. Why not one simple process, transcending, that will enable the entire structure of mathematics education, as we know it, to work? We need to understand and teach heuristics, to establish correct schemas and bases of belief, to identify and correct intuitive biases, to teach informal mathematics as well as formal mathematics. Many of us do. But we also need to teach the ability to transcend. Only then will all our other efforts bear fruit.

At MIU, in my experience, we are enjoying the fruit.

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